

STRONG LAWS FOR RECURRENCE QUANTIFICATION ANALYSIS

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ABSTRACT. The recurrence rate and determinism are two of the basic complexity measures studied in the recurrence quantification analysis. In this paper, the recurrence rate and determinism are expressed in terms of the correlation sum, and strong laws of large numbers are given for them.

1. INTRODUCTION

The notion of recurrence is one of the fundamental notions in the theory of dynamical systems. Recurrence plots, introduced by Eckmann, Kamphorst and Ruelle [4] in 1987, provide a powerful tool for recurrence visualization. The *recurrence plot* of the trajectory x_0, x_1, \dots, x_{n-1} of a point $x = x_0$ is a black-and-white image with a pixel (i, j) being black if and only if the trajectory at time j recurs to the state at time i ; that is, the points x_i, x_j are close to each other. The recurrence plot provides a two-dimensional representation of an (arbitrary-dimensional) dynamical system.

The quantitative study of recurrence plots, called *recurrence quantification analysis (RQA)*, was initiated by Zbilut and Webber in [18], where the authors introduced several measures of complexity based on the recurrence plot. Among them, the recurrence rate RR and the determinism DET are probably the most important and widely used ones. Their definitions are based on *diagonal lines* (that is, segments of black points parallel to the main diagonal), which correspond to recurrences of parts of the trajectory.

Since the seminal paper [18], new RQA tools, quantities and modifications were introduced and recurrence quantification has been applied in many areas of science, cf. [12] and [10], among others.

Despite its wide use, theoretical properties of recurrence measures were studied rarely. Asymptotic properties of RQA characteristics were studied e.g. in [6, 3, 5, 16, 19]. The correlation sum, tightly connected with the recurrence rate, as well as derived quantities such as the correlation integral, correlation dimension and correlation entropy, were studied extensively, cf. [9]. One of the fundamental results, namely the strong law for correlation sums of ergodic processes, was proved (by different methods and under different conditions) in [14, 13, 1, 15, 11]. It states that, for a separable metric space (Z, d) and a μ -ergodic dynamical system on it, the correlation sum of the trajectories of almost every point $x \in Z$ with every (up to countably many) $r > 0$ converges to the correlation integral

$$(1) \quad \lim_{n \rightarrow \infty} C(x, n, r) = c(r), \quad \text{where} \quad c(r) = \mu \times \mu \{(x, y) : d(x, y) \leq r\}$$

and $C(x, n, r) = (1/n^2) \cdot \text{card}\{(i, j) : 0 \leq i, j < n, d(x_i, x_j) \leq r\}$. Recall that the correlation integral $c(r)$ is just the probability $P\{d(X, Y) \leq r\}$ that two independent random variables X, Y with distribution μ are r -close. It is worth noting that the correlation sum, which measures the level of dependence in a trajectory, asymptotically turns into the probability of closeness of two *independent* random variables.

The main purpose of the present paper is to study asymptotic properties of RQA characteristics for ergodic processes. We start with a proof of a simple formula giving an expression of the recurrence rate via correlation sums, see Proposition 1:

$$(2) \quad \text{RR}_k^m = k \cdot C_k^m - (k-1) \cdot C_{k+1}^m,$$

where m is the embedding dimension, k is the prediction horizon and C_k^m and RR_k^m denote the correlation sum and recurrence rate, respectively; for the corresponding definitions see Section 2. The relationship (2)

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directly permits to express the determinism DET in terms of the correlation sums

$$(3) \quad \text{DET}_k^m = \frac{\text{RR}_k^m}{\text{RR}_1^m} = \frac{k \cdot C_k^m - (k-1) \cdot C_{k+1}^m}{C_1^m}.$$

The bridging formulas (2) and (3) enable us to derive strong laws of large numbers for the recurrence rate and determinism from that for the correlation sum, see Theorems 4 and 5. To this end, however, we need to generalize (1) to the case when d is a pseudometric on Z , rather than a metric. For pseudometrics induced by Borel maps, this problem was studied in [15, Theorem 2]. In the general case, (1) was proved by Manning and Simon [11]; for details, see Theorem 19 in Section 7.

We apply the strong laws to iid processes, Markov chains and autoregressive processes, and derive explicit formulas for the recurrence integral, asymptotic determinism and mean diagonal line length of these processes; see Table 1 and Section 4. On simulated data we demonstrate the speed of convergence of RQA quantities when the length n of (the beginning of) the trajectory goes to infinity.

	IID	Markov chain
recurrence integral	$\alpha^{k+m-1} [k - (k-1) \alpha]$	$\alpha \beta^{k+m-2} [k - (k-1) \beta]$
asymptotic determinism	$\alpha^{k-1} [k - (k-1) \alpha]$	$\beta^{k-1} [k - (k-1) \beta]$
mean diagonal line length	$k + \alpha/(1 - \alpha)$	$k + \beta/(1 - \beta)$

TABLE 1. Formulas for RQA asymptotics for iid processes and Markov chains; here $\alpha = c(r)$ and $\beta = c_2(r)/c(r)$.

Further, in Section 5 we give an example showing that higher entropy of a process does not necessarily mean smaller (asymptotic) determinism and that an iid process can have higher determinism than a non-iid one with the same one-dimensional marginals. This is a rather unexpected behavior since, in a sense, entropy and determinism are opposite notions.

We also discuss the problem of choosing the distance threshold r . In the literature, the distance threshold is selected such that, for the embedding dimension m , the recurrence rate attains a fixed level. This rule, however, can lead to the existence of the so-called spurious structures in recurrence plots of iid processes, as noted in [17], see also [12, Section 3.2.4]. In Section 6 we show why this happens. For a large embedding dimension m and distance threshold r_m selected by this rule, the determinism $\text{det}_k^m(r_m)$ is close to one even for iid processes. Hence, the appearance of spurious structures is a direct consequence of the selection rule fixing the recurrence rate.

The explicit formula for the asymptotic determinism det_k^m can be stated in terms of conditional probabilities that k or $(k+1)$ consecutive recurrences occur given that one recurrence has occurred; see Theorem 10. Hence, if the process under consideration is a Markov one of order p , then over-embedding to dimension $m \geq p$ leaves the asymptotic determinism unchanged; see Corollary 11. This is demonstrated in Section 4.3 for autoregressive processes. There we discuss possible use of RQA characteristics for estimation of the order of such processes.

The paper is organized as follows. In Section 2 we recall the definitions of RQA measures and we prove (2), see Proposition 1. The strong laws are stated in Section 3 and, in Section 4, they are applied to iid processes, Markov chains and autoregressive processes. Relationship between entropy and asymptotic determinism is discussed in Section 5 and the so-called spurious structures in recurrence plots of iid processes are explained in Section 6. In Section 7 we discuss the strong law for correlation sums on pseudometric spaces.

2. RECURRENCE QUANTIFICATION ANALYSIS (RQA) AND CORRELATION SUMS

In this section we recall the definitions of basic RQA measures and of the correlation sum for (embedded) trajectories of a general S -valued process. To make the notation easier, we write x_h^m, x_h^∞ as a shorthand for $(x_i)_{i=h}^{h+m-1}, (x_i)_{i=h}^\infty$, respectively.

Let $S = (S, \varrho)$ be a metric space. Fix an integer $m \geq 1$ called the *embedding dimension*. Let S^m be the *embedding space* of m -tuples s_0^m equipped with a metric ϱ^m compatible with the product topology. Natural

choices for ϱ^m are the Manhattan (L_1), Euclidean (L_2) or Chebyshev (L_∞) metrics, the latter given by

$$(4) \quad \varrho^m(s_0^m, t_0^m) = \max_{0 \leq j < m} \varrho(s_j, t_j),$$

but in general we do not restrict ϱ^m to be one of these.

Let S^∞ denote the space of all sequences x_0^∞ of points from S . This product space is usually equipped with a metric, say with $\varrho^\infty(x_0^\infty, y_0^\infty) = \sum_i 2^{-i} \cdot \min\{1, \varrho(x_i, y_i)\}$. In practice, however, we know just (finite) beginnings of trajectories x_0^∞ and thus we are not able to compute the distance exactly. That is why we use pseudometrics instead, depending only on the first members of sequences. For an integer $k \geq 1$, called the *prediction horizon*, a pseudometric d_k^m on S^∞ is defined by

$$(5) \quad d_k^m(x_0^\infty, y_0^\infty) = \max_{0 \leq i < k} \varrho^m(x_i^m, y_i^m).$$

For $k = 1$ we write simply d^m instead of d_1^m . Notice that d_k^m depends only on the first $(m + k - 1)$ members of x_0^∞, y_0^∞ .

2.1. RQA measures. Fix a sequence $x = x_0^\infty \in S^\infty$ and consider the embedded trajectory $\tilde{x} = \tilde{x}_0^\infty$, $\tilde{x}_i = x_i^m \in S^m$. Fix also a distance threshold $r \geq 0$. For $i, j \in \mathbb{N}$ (here \mathbb{N} stands for the set of non-negative integers $\{0, 1, \dots\}$) we say that the couple (i, j) is an *r -recurrence* (in the m 'th embedding of the trajectory of x) if

$$d^m(x_i^\infty, x_j^\infty) = \varrho^m(x_i^m, x_j^m) \leq r.$$

The *recurrence plot* of dimension n is a square $n \times n$ matrix of zeros and ones, with the entry at (i, j) ($0 \leq i, j < n$) equal to one if and only if (i, j) is a recurrence. Usually, the recurrence plot is visualized by a black-and-white image, with black pixels representing recurrences. Let us note that to construct the $n \times n$ recurrence plot (in the m 'th embedding) one needs to know only the first $(n + m - 1)$ members x_0^{n+m-1} of x .

Diagonal lines are basic patterns in the recurrence plot. We say that (i, j) is a beginning of a *diagonal line of length $k \geq 1$* in the $n \times n$ recurrence plot if the following are true:

- $0 \leq i, j \leq n - k$;
- $(i + h, j + h)$ is a recurrence for every $0 \leq h < k$;
- either at least one of i, j is equal to 0 or $(i - 1, j - 1)$ is not a recurrence;
- either at least one of $i + k, j + k$ is equal to n or $(i + k, j + k)$ is not a recurrence.

For $0 < i, j < n - k$ this is equivalent to

$$d_k^m(x_i^\infty, x_j^\infty) \leq r, \quad d^m(x_{i-1}^\infty, x_{j-1}^\infty) > r \quad \text{and} \quad d^m(x_{i+k}^\infty, x_{j+k}^\infty) > r.$$

The number of lines of length k in the $n \times n$ recurrence plot is denoted by $L_k^m = L_k^m(x, n, r)$. Notice that the main diagonal line (i.e. the case $i = j$) is not excluded, thus $L_n^m(x, n, r) = 1$; further, $L_k^m(x, n, r) = 0$ for every $k > n$.

Now fix the prediction horizon $k \geq 1$. The *k -recurrence rate* RR_k^m is the percentage of recurrences contained in diagonal lines of length at least k ; that is,

$$(6) \quad RR_k^m = RR_k^m(x, n, r) = \frac{1}{n^2} \sum_{l \geq k} l \cdot L_l^m.$$

The *k -determinism* DET_k^m is the ratio of the k -recurrence rate and 1-recurrence rate

$$(7) \quad DET_k^m = DET_k^m(x, n, r) = \frac{RR_k^m}{RR_1^m}$$

(here and throughout we always assume that the denominator is non-zero; otherwise we leave the corresponding quantity undefined). The *k -average line length* $LAVG_k^m$ is the average length of diagonal lines not shorter than k

$$(8) \quad LAVG_k^m = \frac{RR_k^m}{(1/n^2) \sum_{l \geq k} L_l^m};$$

again, this characteristic depends also on x, n, r . For the definitions of other RQA characteristics, such as the (Shannon) entropy of diagonal line length, trend or measures based on vertical lines, see e.g. [12].

2.2. Correlation sum. Tightly connected with the recurrence rate is the notion of correlation sum, studied by Grassberger and Procaccia [8, 7] in relation to the correlation dimension. For a sequence $x = x_0^\infty \in S^\infty$, the embedding dimension m , the prediction horizon k , the distance threshold $r \geq 0$ and $n \geq 1$, the *correlation sum* is defined by

$$(9) \quad C_k^m = C_k^m(x, n, r) = \frac{1}{n^2} \text{card}\{(i, j) : 0 \leq i, j \leq (n - k), d_k^m(x_i^\infty, x_j^\infty) \leq r\}.$$

Here, as above, the quantity depends only on the beginning x_0^{n+m-1} of x . C_k^m measures the relative frequency of recurrences (in the m 'th embedding) followed by at least $(k - 1)$ other recurrences. Since, in a diagonal line of length $l \geq k$, just the first $(l - k + 1)$ points are followed by $(k - 1)$ other recurrences, it immediately follows that

$$(10) \quad C_k^m = \frac{1}{n^2} \sum_{l \geq k} (l - k + 1) L_l^m$$

for every $m, k \geq 1$. Comparison with (6) gives the next statement.

Proposition 1. *For $m, k \geq 1$,*

$$\text{RR}_k^m = k \cdot C_k^m - (k - 1) \cdot C_{k+1}^m.$$

Proof. By (10) and (6) we have

$$\begin{aligned} n^2 \text{RR}_k^m &= k \left[n^2 C_k^m - \sum_{l \geq k+1} (l - k + 1) L_l^m \right] + \sum_{l \geq k+1} l L_l^m \\ &= k n^2 C_k^m - (k - 1) \sum_{l \geq k+1} (l - k) L_l^m \\ &= k n^2 C_k^m - (k - 1) n^2 C_{k+1}^m, \end{aligned}$$

from which the assertion immediately follows. \square

Validity of the previous relation can be also seen from the following picture

$$\cdots \circ \underbrace{\bullet \cdots \bullet}_a \underbrace{\bullet \cdots \bullet}_{b: |b|=k} \circ \cdots$$

of a diagonal line of length $l = |a| + |b| \geq k$. The a -dots are counted in the k -recurrence rate as well as in both the k and $(k + 1)$ -correlation sum. On the other hand, all of the b -dots are counted in the k -recurrence rate, but only the first one is counted in the k -correlation sum and none in the $(k + 1)$ -correlation sum. This gives $\text{RR}_k^m = k(C_k^m - C_{k+1}^m) + C_{k+1}^m$, which is equivalent to the formula from Proposition 1.

As a corollary of Proposition 1 we can immediately obtain a formula for the determinism in terms of correlation sums. Since (10) gives $(1/n^2) \sum_{l \geq k} L_l^m = C_k^m - C_{k+1}^m$, we also obtain that

$$(11) \quad \text{LAVG}_k^m = k + \frac{C_{k+1}^m}{C_k^m - C_{k+1}^m}.$$

As was noted by many authors, if the metric ϱ^m in the embedding space is the Chebyshev one (see (4)), the embedded recurrence quantities can be expressed in terms of the non-embedded ones. Let us formulate this as a lemma; there, L_k, C_k, RR_k stand for $L_k^1, C_k^1, \text{RR}_k^1$, respectively

Lemma 2. *Let $m, k \geq 1$ and ϱ^m be given by (4). Then*

$$L_k^m(x, n, r) = L_h(x, n', r), \quad C_k^m(x, n, r) = \left(\frac{n'}{n}\right)^2 \cdot C_h(x, n', r)$$

and

$$\text{RR}_k^m(x, n, r) = \left(\frac{n'}{n}\right)^2 \cdot [\text{RR}_h(x, n', r) - (m - 1) \cdot (C_h(x, n', r) - C_{h+1}(x, n', r))],$$

where $h = k + m - 1$ and $n' = n + m - 1$.

Proof. By (4), $d_k^m(x_i^\infty, x_j^\infty) \leq r$ if and only if $\varrho(x_{i+l}, x_{j+l}) \leq r$ for every $0 \leq l < h$. Thus the first equality is an immediate consequence of the definition of diagonal lines and the second one follows from (10). Further, (6) gives

$$\begin{aligned} n^2 \text{RR}_k^m(x, n, r) &= \sum_{l' \geq h} (l' - (m-1)) \cdot L_h(x, n', r) \\ &= (n')^2 \cdot \text{RR}_h(x, n', r) - (m-1) \cdot \sum_{l' \geq h} L_h(x, n', r). \end{aligned}$$

Hence, using (10), also the third formula is proved. \square

3. STRONG LAWS FOR RQA

Here, among other results, we formulate and prove strong laws of large numbers for the recurrence rate and determinism. First, the necessary notions and results are summarized. By a *space* we always mean a topological space.

3.1. Preliminaries. Let Z be a space and \mathcal{B}_Z be the Borel σ -algebra on Z . A (*measure-theoretical*) *dynamical system* is a quadruple $(Z, \mathcal{B}_Z, \mu, T)$, where μ is a probability measure on (Z, \mathcal{B}_Z) and $T : Z \rightarrow Z$ is a (Borel) measurable map which preserves μ , that is, $\mu(T^{-1}(B)) = \mu(B)$ for every $B \in \mathcal{B}_Z$. A set $B \in \mathcal{B}_Z$ is said to be *T-invariant* if $T^{-1}(B) = B$. We say that T is μ -*ergodic* or that μ is *T-ergodic* if $\mu(B) \in \{0, 1\}$ for every T -invariant set B . For $n \in \mathbb{N}$, the n -th (*forward*) *iterate* T^n of T is defined recursively by $T^0 = \text{id}_Z$ and $T^{n+1} = T \circ T^n$. For $m, n \geq 0$ and $x \in Z$ we write T_n^m and $T_n^m(x)$ instead of $(T^i)_{i=n}^{n+m-1}$ and $(T^i(x))_{i=n}^{n+m-1}$, respectively.

Let S be a space with the Borel σ -algebra \mathcal{B}_S . On the product space S^∞ , the Borel σ -algebra is denoted by \mathcal{B}_S^∞ . An S -valued (*discrete time*) *stochastic process* is a sequence $X = X_0^\infty$ of random variables $X_n : \Omega \rightarrow S$ ($n \in \mathbb{N}$) defined on a probability space (Ω, \mathcal{B}, P) . The *distribution of the process* X is the measure $\mu = \mu_X$ on $(S^\infty, \mathcal{B}_S^\infty)$ defined by $\mu(F) = P\{X_0^\infty \in F\}$.

The (*left*) *shift* on S^∞ is the (continuous) map $T : S^\infty \rightarrow S^\infty$ defined by

$$T(x_0^\infty) = y_0^\infty, \quad \text{where } y_n = x_{n+1} \text{ for every } n \in \mathbb{N}.$$

Let $\pi : S^\infty \rightarrow S$ denote the projection onto the zeroth coordinate, that is, $\pi(x_0^\infty) = x_0$. If X is a stochastic process with distribution μ , then the shift T together with the projection π and the measure μ form the *Kolmogorov representation* of the process X . From now on we always assume that X is directly given by its Kolmogorov representation, that is,

$$(\Omega, \mathcal{B}, P) = (S^\infty, \mathcal{B}_S^\infty, \mu) \quad \text{and} \quad X_n = \pi \circ T^n.$$

A process X is (*strictly*) *stationary* if its distribution μ is T -invariant. The *marginal* of a stationary process X_0^∞ is the distribution of X_0 . A process X is *ergodic* if every T -invariant event has probability either 0 or 1. Thus, a process X is stationary and ergodic if and only if the dynamical system $(S^\infty, \mathcal{B}_S^\infty, \mu, T)$ is ergodic.

3.2. Strong law for correlation sum. For a Borel measure μ on S^∞ , $m, k \geq 1$ and $r \geq 0$ define the *correlation integral* $c_k^m(r)$ by

$$(12) \quad c_k^m(r) = \mu \times \mu \{(x, y) : d_k^m(x, y) \leq r\}.$$

If μ is the distribution of a process X_0^∞ , then $c_k^m(r)$ is the probability that, for two independent random vectors Y_0^{k+m-1}, Z_0^{k+m-1} with the distribution equal to that of X_0^{k+m-1} , every Y_i^m, Z_i^m ($i < k$) are r -close according to ϱ^m :

$$(13) \quad c_k^m(r) = \mu \{\varrho^m(Y_i^m, Z_i^m) \leq r \text{ for every } 0 \leq i < k\}.$$

The following theorem, the proof of which is postponed to Section 7, follows from [11].

Theorem 3. *Let S be a separable metric space, X be an S -valued ergodic stationary process with distribution μ and $m, k \geq 1$ be integers. Then for μ -a.e. trajectory $x \in S^\infty$ of X and for every $r > 0$*

$$(14) \quad \lim_{n \rightarrow \infty} C_k^m(x, n, r) = c_k^m(r) > 0$$

provided c_k^m is continuous at r .

Notice that c_k^m is right continuous and non-decreasing, so it has at most countably many discontinuities; it is continuous at r if and only if $\mu \times \mu\{(x, y) : d_k^m(x, y) = r\}$ is zero. Further, the convergence in (13) is uniform over r on any compact interval on which c_k^m is continuous.

3.3. Strong laws for RQA. The purpose of this section is to show that the basic RQA characteristics converge almost surely to constants, which depend only on the distribution μ of the (ergodic) process and on the distance threshold r . To formulate the results, we introduce the *recurrence integral* rr_k^m , *asymptotic determinism* det_k^m and *mean diagonal line length* lavg_k^m for every r by

$$(15) \quad \begin{aligned} \text{rr}_k^m(r) &= k \cdot c_k^m(r) - (k-1) \cdot c_{k+1}^m(r), \\ \text{det}_k^m(r) &= \frac{\text{rr}_k^m(r)}{\text{rr}_1^m(r)} \quad \text{and} \\ \text{lavg}_k^m(r) &= k + \frac{c_{k+1}^m(r)}{c_k^m(r) - c_{k+1}^m(r)}; \end{aligned}$$

if k is such that $c_k^m(r) = c_{k+1}^m(r) > 0$ we put $\text{lavg}_k^m(r) = \infty$. Thus all the quantities are defined for every $r > 0$.

Proposition 1 and Theorem 3 immediately give the following theorem.

Theorem 4 (Strong law for recurrence rate). *Under the assumptions of Theorem 3, for μ -a.e. $x \in S^\infty$ and for every (up to countably many) $r > 0$,*

$$\lim_{n \rightarrow \infty} \text{RR}_k^m(x, n, r) = \text{rr}_k^m(r).$$

Theorem 5 (Strong laws for DET and LAVG). *Under the assumptions of Theorem 3, for μ -a.e. $x \in S^\infty$ and for every (up to countably many) $r > 0$,*

$$\lim_{n \rightarrow \infty} \text{DET}_k^m(x, n, r) = \text{det}_k^m(r) \quad \text{and} \quad \lim_{n \rightarrow \infty} \text{LAVG}_k^m(x, n, r) = \text{lavg}_k^m(r).$$

Proof. The statements follow since a.e.-convergence is preserved by elementary arithmetic operations provided that, for division, the numerator or denominator is non-zero. \square

Remark 6. Theorem 3 can be trivially used to derive strong law also for another RQA quantity called the *k-ratio* defined by $\text{RATIO}_k^m = \text{DET}_k^m / \text{RR}_1^m$. Further, for the *maximal diagonal line length* LMAX^m defined by

$$\text{LMAX}^m = \text{LMAX}^m(x, n, r) = \max\{l < n : L_l^m(x, n, r) > 0\}$$

using Birkhoff ergodic theorem one can easily show that, under the assumptions of Theorem 3,

$$\lim_{n \rightarrow \infty} \text{LMAX}^m(x, n, r) = \infty$$

for μ -a.e. $x \in S^\infty$ and for every $r > 0$. As a corollary we immediately have that the reciprocal value $\text{DIV}^m = 1/\text{LMAX}^m$ called the *divergence* converges almost surely to zero.

Remark 7. Recurrence measures as well as correlation sums are often defined using strict inequalities $d_k^m(x, y) < r$, and/or with excluding the main diagonal $i = j$. Clearly, the latter has no effect on asymptotic properties, that is, Theorems 3–5 remain true also in this case. When one uses strict inequalities, then again the results are valid provided strict inequality is used also in the definition (12) of the correlation integral. The relationship between this new “open” correlation integral and the used “closed” one is straightforward, see [13, Remark 2.2].

Remark 8. As can be seen from Theorem 19, Theorem 3 is valid with d_k^m replaced by any separable Borel pseudometric d on S^∞ . For example, d can be defined via order patterns (cf. [2]): $d(x_0^\infty, y_0^\infty) = 1$ if x_0^m, y_0^m have the same order pattern, $d(x_0^\infty, y_0^\infty) = 0$ otherwise. In this way we obtain strong laws for RQA characteristics based on order patterns recurrence plots.

As for “empirical” RQA quantities (see Lemma 2), the dependence of asymptotic ones on the embedding dimension m is straightforward provided the maximum metric is used.

Lemma 9. *Let $m, k \geq 1$, $r \geq 0$ and ϱ^m be given by (4). Then*

$$c_k^m = c_h, \quad \text{rr}_k^m = \text{rr}_h - (m-1)(c_h - c_{h+1}) \quad \text{and} \quad \text{lav}_k^m = \text{lav}_h - (m-1),$$

where $h = k + m - 1$.

Proof. The first equality follows from (12) and the definition (5) of d_k^m . The others are then consequences of (15). \square

3.4. Asymptotic determinism via conditional probabilities. Here we assume (4). For $h, l \geq 1$ and $r > 0$ define the *conditional correlation integral* by

$$c_{l|h}(r) = \frac{c_{h+l}(r)}{c_h(r)} = \frac{\mu \times \mu\{(y, z) : d_{h+l}(y, z) \leq r\}}{\mu \times \mu\{(y, z) : d_h(y, z) \leq r\}}.$$

Particularly, if μ is the distribution of an ergodic stationary process X_0^∞ and Y_0^{h+l}, Z_0^{h+l} are independent random vectors with the distribution equal to that of X_0^{h+l} , then $c_{l|h}(r)$ is the conditional probability

$$c_{l|h}(r) = \mu \{ \varrho^l(Y_h^l, Z_h^l) \leq r \mid \varrho^h(Y_0^h, Z_0^h) \leq r \}.$$

Thus, $c_{l|h}(r)$ is the probability that h consecutive recurrences are followed by at least l other ones. In view of this we have the following interesting expression of asymptotic determinism in terms of conditional probabilities.

Theorem 10. *Under (4), the asymptotic determinism can be expressed via a linear combination of conditional correlation integrals*

$$\det_k^m = k \cdot c_{k-1|m} - (k-1) \cdot c_{k|m}.$$

Consider now the special case of (ergodic stationary) Markov processes of order $p \geq 1$. Then for every $m \geq p$ one has $c_{l|m} = c_{l|p}$. That is, over-embedding has no effect on the asymptotic determinism.

Corollary 11. *For every (ergodic stationary) Markov process of order $p \geq 1$ and for every $m \geq p$, $\det_k^m = \det_k^p$.*

4. ASYMPTOTIC RQA MEASURES FOR SOME PROCESSES

Now we present some applications of the asymptotic results obtained in the previous section. We assume that $S = (S, \varrho)$ is a separable metric space and S^∞ is equipped with the pseudometric d_k^m given by (5), where m is the embedding dimension, k is the prediction horizon and the embedding metric ϱ^m is given by (4). We also assume that X_0^∞ is (a Kolmogorov representation of) an ergodic stationary S -valued process. In the following we derive explicit formulas for asymptotic RQA measures for some classes of processes. To make the paper self-contained we include here also the proofs, though the results (at least for correlation integrals) are known. The convergence is demonstrated by simulation studies. We start with the simplest case of iid processes.

4.1. IID processes.

Proposition 12. *Let X_0^∞ be an iid process. Then, for $m, k \geq 1$ and $r \geq 0$,*

$$c_k^m(r) = \alpha^{m+k-1}, \quad \text{where } \alpha = c(r).$$

Hence

$$\det_k^m(r) = \alpha^{k-1} [k - (k-1)\alpha] \quad \text{and} \quad \text{lav}_k^m(r) = k + \frac{\alpha}{1-\alpha}$$

do not depend on the embedding dimension m .

Proof. By Lemma 9 we may assume that $m = 1$. Let Y_0^k, Z_0^k be independent random vectors with the distribution equal to that of X_0^k . Then, for every $r > 0$,

$$\begin{aligned} c_k(r) &= \mu\{\varrho_k(Y_0^k, Z_0^k) \leq r\} = \mu\{\varrho(Y_i, Z_i) \leq r \text{ for every } 0 \leq i < k\} \\ &= \prod_{0 \leq i < k} \mu\{\varrho(Y_i, Z_i) \leq r\} = [c(r)]^k. \end{aligned}$$

Thus the first statement is proved. The rest follows from the definitions (15) of \det_k^m and lav_k^m . \square

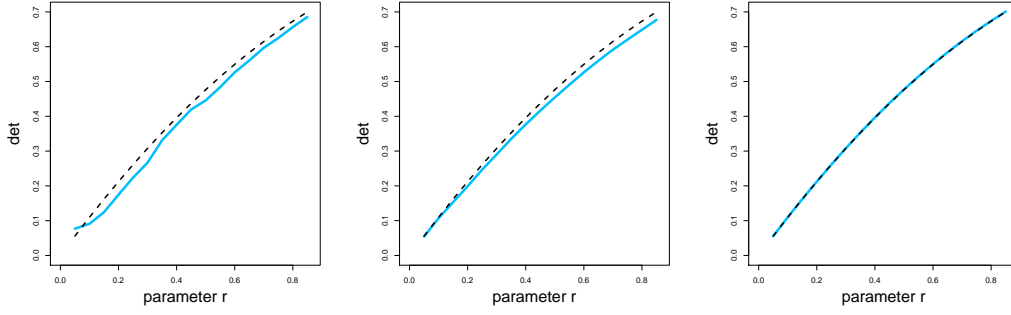


FIGURE 1. Convergence of the empirical determinism (solid line) to the asymptotic determinism (dashed line) for an iid process with distribution $N(0, 1)$; $n = 100, 1000, 10000$.

For example, the asymptotic determinism of a Gaussian iid process with variance σ^2 is

$$\det_k^m(r) = 2k [2\Phi(r') - 1]^{k-1} \cdot \left[1 - \frac{1}{2k} - \Phi(r')\right], \quad \text{where } r' = \frac{r}{\sqrt{2}\sigma}$$

and Φ is the distribution function of the standard normal distribution. To see this, use that for iid Gaussian random variables Y, Z with variance σ^2 , $Y - Z \sim N(0, 2\sigma^2)$ and so $c(r) = \mu\{|Y - Z| \leq r\} = 2\Phi(r') - 1$.

Figure 1 illustrates the convergence of the empirical determinism (with $m = 1$ and $k = 2$) to the asymptotic one for a Gaussian iid process.

4.2. Markov chains. Let $S = \{0, 1, \dots, q-1\}$ be a finite space equipped with the discrete metric ϱ (that is, $\varrho(x, y) = 1$ if $x \neq y$ and $\varrho(x, y) = 0$ for $x = y$). Consider an S -valued Markov chain X_0^∞ with the transition matrix $P = (p_{st})_{s,t=0}^{q-1}$ and the stationary distribution $\pi = (\pi_0, \dots, \pi_{q-1})'$. Recall that $\pi'P = \pi'$ and that X_0^∞ is ergodic if and only if the matrix P is transitive or, equivalently, the probability of the transition from any state s to any state t in a finite time is non-zero. The formulas for asymptotic values of RQA characteristics of Markov chains are given in the following proposition. As in the iid case, also here we can see that both the determinism and mean diagonal line length do not depend on the embedding dimension. (Notice that, in this discrete setting, only the distance threshold r less than 1 needs to be considered and, for $0 \leq r < 1$, RQA quantities do not depend on r .)

Proposition 13. *Let X_0^∞ be a finite-valued Markov chain with the transition matrix P and the stationary distribution π . Then, for $r \in [0, 1)$,*

$$c_k^m(r) = \alpha\beta^{k+m-2}, \quad \det_k^m(r) = \beta^{k-1} [k - (k-1)\beta] \quad \text{and} \quad \text{lav}_k^m(r) = k + \frac{\beta}{1-\beta}$$

where $\alpha = \pi'\pi$ and $\beta = (\pi'\text{diag}(PP')\pi)/\alpha$.

Proof. Only the equality for $c_k^m(r)$ needs a proof since the other two follow from (15). We may assume that $m = 1$. Let Y_0^k, Z_0^k be independent random vectors with the distribution equal to that of X_0^k . Fix $r < 1$ and put $\alpha = c(r)$; then $\alpha = \mu\{d(Y_0, Z_0) \leq r\} = \sum_s \mu\{Y_0 = Z_0 = s\} = \pi'\pi$.

If $k = 1$ we are done. So assume that $k \geq 2$ and put $\beta = c_{1|1}(r) = \mu\{d(Y_1, Z_1) \leq r \mid d(Y_0, Z_0) \leq r\}$. Then

$$\begin{aligned} \beta &= \frac{1}{\alpha} \mu\{Y_0 = Z_0, Y_1 = Z_1\} = \frac{1}{\alpha} \sum_{s,t} \mu\{Y_0 = Z_0 = s, Y_1 = Z_1 = t\} \\ &= \frac{1}{\alpha} \sum_{s,t} \mu\{X_0 = s, X_1 = t\}^2 = \frac{1}{\alpha} \sum_{s,t} \pi_s^2 \cdot p_{st}^2 = \frac{1}{\alpha} (\pi' \text{diag}(PP') \pi). \end{aligned}$$

Since X_0^∞ is a stationary Markov chain, we obtain

$$\begin{aligned} c_k(r) &= \mu\{d_k(Y_0^k, Z_0^k) \leq r\} = \mu\{Y_i = Z_i \ \forall 0 \leq i < k\} \\ &= \mu\{Y_{k-1} = Z_{k-1} \mid Y_i = Z_i \ \forall 0 \leq i < k-1\} \cdot c_{k-1}(r) = \beta c_{k-1}(r). \end{aligned}$$

Now a simple induction gives the desired result. \square

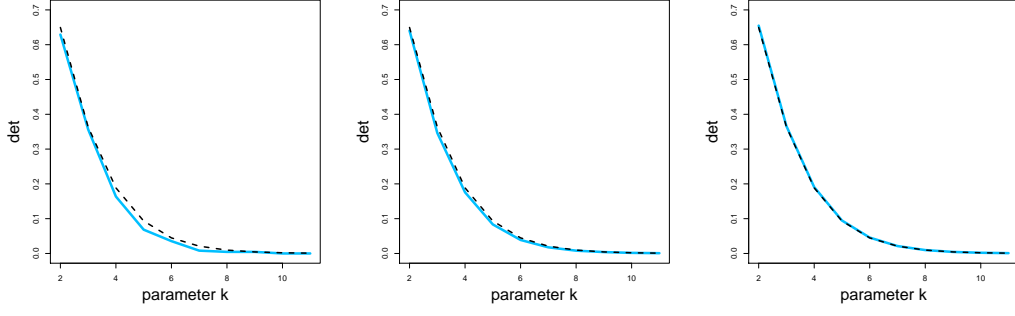


FIGURE 2. Convergence of the empirical determinism (solid line) to the asymptotic determinism (dashed line) for a 3-state Markov chain with the transition matrix P ; $n = 100, 1000, 10000$.

Figure 2 depicts the convergence of the empirical determinism (with $m = 1$) to the asymptotic one for a 3-state Markov chain with the (randomly selected) transition matrix $P = \begin{pmatrix} 0.362 & 0.438 & 0.200 \\ 0.484 & 0.447 & 0.069 \\ 0.120 & 0.503 & 0.377 \end{pmatrix}$.

4.3. Autoregressive processes. Next we consider asymptotic RQA characteristics of a (stationary) autoregressive process $X_0^\infty \sim AR(p)$ of order $p \geq 1$ with coefficients θ_i ($i = 1, \dots, p$) and with Gaussian zero mean noise ε_0^∞ of variance σ^2 . It is given by $X_n = \theta_1 \cdot X_{n-1} + \theta_2 \cdot X_{n-2} + \dots + \theta_p \cdot X_{n-p} + \varepsilon_n$.

Proposition 14. *Let X_0^∞ be an (ergodic stationary) autoregressive process $AR(p)$ with coefficients $\theta_1, \dots, \theta_p$ and Gaussian $WN(0, \sigma^2)$. Let $r > 0$, $m, k \geq 1$ and $h = k + m - 1$. Then*

$$c_k^m(r) = \mu\{Y_0^h \in [-r, r]^h\},$$

where $Y_0^h \sim N(0, \Sigma)$ with Σ being the $h \times h$ autocovariance matrix of an $AR(p)$ process with coefficients $\theta_1, \dots, \theta_p$ and Gaussian $WN(0, 2\sigma^2)$.

Proof. Since the difference of two independent AR processes with the same parameters $\theta_1, \dots, \theta_p, \sigma^2$ is an $AR(p)$ process with the same coefficients and noise variance $2\sigma^2$, the statement immediately follows from (13). \square

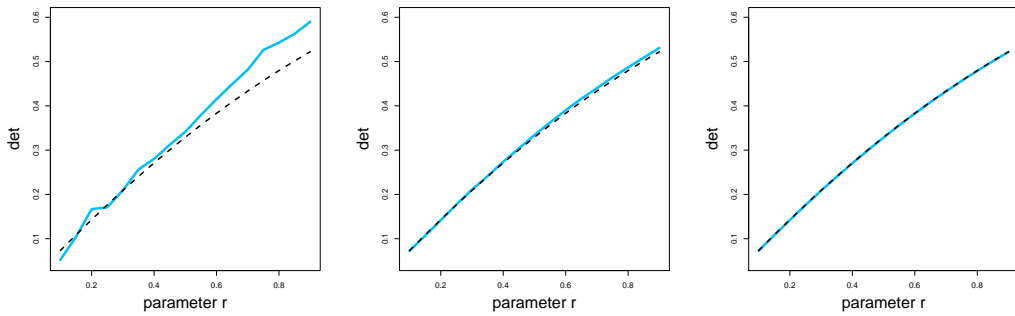


FIGURE 3. Convergence of the empirical determinism (solid line) to the asymptotic determinism (dashed line) for an $AR(3)$ process with parameters $a_1 = 0.25, a_2 = 0.4, a_3 = 0.3$ and $\sigma^2 = 1.5$; $n = 100, 1000, 10000$.

The convergence of the empirical determinism DET_2^1 to the asymptotic one for an AR process is exhibited in Figure 3.

Corollary 11 implies that over-embedding of an $AR(p)$ process into dimension $m > p$ leaves the determinism unchanged. Thus, the asymptotic determinism can be used to estimate (from below) the order of an autoregressive process. This is demonstrated in Table 2 on an $AR(3)$ process. There one can see that embedding into dimension 4 or 5 gives the determinism equal to that for dimension 3, but the determinisms for $m = 1, 2$ are smaller. Thus one can conclude that the order of the process is at least 3.

$\begin{array}{c} \text{m} \\ \backslash \\ k \end{array}$	1	2	3	4	5
2	0.638 (0.009)	0.725 (0.008)	0.759 (0.008)	0.760 (0.008)	0.760 (0.009)
3	0.407 (0.010)	0.491 (0.011)	0.514 (0.012)	0.515 (0.013)	0.516 (0.014)
4	0.258 (0.010)	0.312 (0.011)	0.327 (0.012)	0.328 (0.013)	0.329 (0.014)
5	0.158 (0.008)	0.191 (0.010)	0.201 (0.011)	0.201 (0.012)	0.202 (0.013)

TABLE 2. Determinisms for $AR(3)$ process with $a_1 = 0.25, a_2 = 0.4, a_3 = 0.3$ and white noise's variance 1.5; here $n = 2500$ and $r = \sqrt{1.5}$. Average determinisms with standard errors in parentheses obtained by a Monte Carlo simulation of size 1000.

5. KOLMOGOROV ENTROPY AND ASYMPTOTIC DETERMINISM

In the following three examples we demonstrate that behavior of the RQA determinism can sometimes be counterintuitive. First we show that the determinism of an iid process can be higher than that of a non-iid one with the same marginal. In the second example it is shown that higher entropy does not necessarily mean smaller determinism. Finally, a Markov chain indistinguishable (from the RQA point of view) from an iid process is constructed.

Example 15 (Determinism of iid and non-iid processes). Fix $0 < a, b < 1$ and consider a 01-valued Markov chain X_0^∞ with the transition matrix $P = (p_{st})_{s,t=0}^1$ such that $p_{00} = a, p_{11} = b$. Then X_0^∞ is ergodic and the stationary distribution of it is given by $\pi = \left(\frac{1-b}{2-a-b}, \frac{1-a}{2-a-b} \right)'$. Fix any $r \in [0, 1)$. By Proposition 13,

$$\det_k^m(r)^{\text{Markov}} = k\beta^{k-1} - (k-1)\beta^k, \quad \text{where } \beta = c_{1|1}(r) = \frac{(1-a)^2 \cdot [b^2 + (1-b)^2] + (1-b)^2 \cdot [a^2 + (1-a)^2]}{(1-a)^2 + (1-b)^2}.$$

On the other hand, for a 01-valued iid process with the same marginal π , Proposition 12 gives

$$\det_k^m(r)^{\text{iid}} = k\alpha^{k-1} - (k-1)\alpha^k, \quad \text{where } \alpha = \frac{(1-a)^2 + (1-b)^2}{(2-a-b)^2}.$$

If we take $a = 3/5$ and $b = 1/5$, then $\alpha > \beta$. Since the function $x \mapsto kx^{k-1} - (k-1)x^k$ is increasing on $[0, 1]$, we have that $\det_k^m(r)^{\text{Markov}} > \det_k^m(r)^{\text{iid}}$ for any $m, k \geq 1$.

Example 16 (Determinism and entropy). The previous example also shows that higher entropy does not necessarily mean smaller determinism. In fact, the entropy of an iid process is strictly larger than that of any stationary non-iid process with the same marginal. In this simple case the entropies can be calculated analytically, since the entropy of an iid process is $h^{\text{iid}} = -\sum_s \pi_s \log \pi_s$ and the entropy of the Markov chain is $h^{\text{Markov}} = -\sum_{st} \pi_s p_{st} \log p_{st}$. See also Figure 4 for an illustration of this phenomenon.

Example 17 (Indistinguishable Markov chain and iid process). In the 2-state Markov chain considered in Example 15, fix $b = 1/5$ and, for given a , denote by α_a, β_a the corresponding correlation integrals $c_1(r), c_{1|1}(r)$, respectively. Since $\alpha_{1/2} < \beta_{1/2}$ and $\alpha_{3/5} > \beta_{3/5}$, there is $a \in (1/2, 3/5)$ with $\alpha_a = \beta_a$. For this particular (non-iid) Markov chain X_0^∞ , the probability of finding a diagonal line of length k (in the infinite recurrence plot) is the same as that for an iid process Y_0^∞ with the same marginal. Hence, no RQA measure based on diagonal lines can distinguish between X_0^∞, Y_0^∞ .

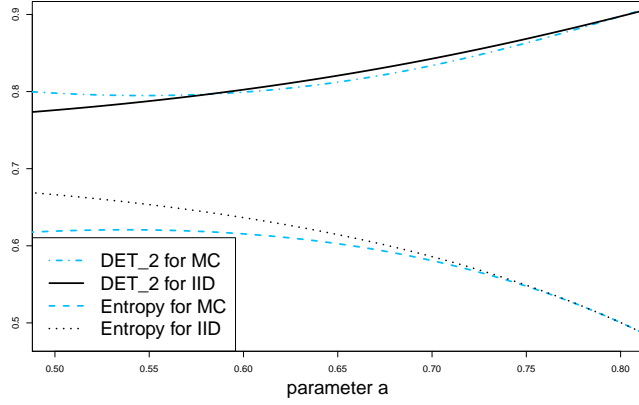


FIGURE 4. Entropy versus determinism \det_2^1 for a 2-state Markov chain (MC) with $b = 1/5$ and an iid process with the same marginal.

6. SPURIOUS STRUCTURES

In [17], see also [12, Section 3.2.4], it was pointed out that, for iid processes, over-embedding leads to existence of spurious structures in recurrence plots. The appearance of spurious structures is illustrated in Figure 5. The left panel depicts the “usual” recurrence plot of an iid process for the embedding dimension $m = 1$. On the right panel there is the recurrence plot for the embedding dimension $m = 250$. It contains long diagonal lines, which would suggest that the process should be well predictable.

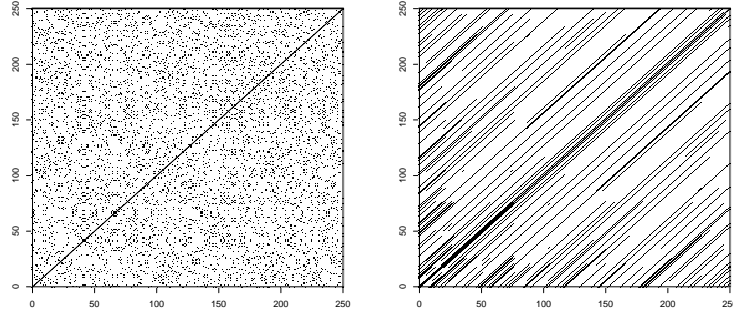


FIGURE 5. The effect of equal recurrence rates for embedding dimensions $m = 1$ and 250; uniform iid data.

Proposition 12 enables us to explain why this happens. In fact, this is due to a special choice of the distance threshold r , which selects such $r = r_m$ that the recurrence rate $\text{RR}_1^m(r_m)$ is fixed to a predetermined level. As the following proposition demonstrates, this selection rule leads to the determinism close to one and average diagonal line length arbitrarily high for large embedding dimensions.

Proposition 18. *Let X_0^∞ be an iid process. Let $\theta > 0$ and let $r_m > 0$ ($m \in \mathbb{N}$) be such that all the recurrence rates $\text{rr}_1^m(r_m)$ are equal to θ . Then, for $k \geq 1$,*

$$\lim_{m \rightarrow \infty} \det_k^m(r_m) = 1 \quad \text{and} \quad \lim_{m \rightarrow \infty} \text{avg}_k^m(r_m) = \infty.$$

Proof. For $m \geq 1$ put $\alpha_m = c(r_m)$. Then, by the assumption and Proposition 12, $\theta = \text{rr}_1^m(r_m) = (\alpha_m)^m$ for every m ; thus $\alpha_m = \theta^{1/m} \rightarrow 1$ for $m \rightarrow \infty$. Using Proposition 12 we obtain that, for every $k \geq 1$, $\text{rr}_k^m(r_m) = (\alpha_m)^{m+k-1} \cdot [k - (k-1)\alpha_m] \rightarrow \theta$ for $m \rightarrow \infty$, and so $\lim_m \det_k^m(r_m) = 1$ and $\lim_m \text{avg}_k^m(r_m) = \infty$. \square

Hence, appearance of the spurious structures for iid processes is an artefact of this particular selection rule for density thresholds. The artificial “predictability” which appears on the right panel of Figure 5 is due to the distance threshold r , which is several times higher than the standard deviation of the process. Different selection rule, which chooses r independently of the embedding dimension, leaves the determinisms $\det_k^m(r)$ and mean diagonal line lengths $\text{lav}_k^m(r)$ constant for $m \rightarrow \infty$, as one expects for iid processes.

7. STRONG LAW FOR CORRELATION SUMS ON PSEUDOMETRIC SPACES

Here we give a proof of Theorem 3, based on the strong law for correlation sums on pseudometric spaces; see Theorem 19 below. Recall that, for a (topological) space Z , a map $d : Z \times Z \rightarrow \mathbb{R}^+$ is a *pseudometric* on Z if $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in Z$. A pseudometric d is *separable* if the topology generated by it is separable. We say that d is a *Borel (continuous) pseudometric* on Z if it is a pseudometric which is Borel (continuous) w.r.t. the product topology on $Z \times Z$. Notice that a continuous pseudometric on a separable space is automatically separable.

If d is a pseudometric on Z , $B_d(x, r)$ and $S_d(x, r)$ denote the (closed) d -ball and d -sphere with radius r centered at x , respectively. Notice that if d is Borel then d -balls and d -spheres are Borel sets in Z ; to see it, use that $B_d(x, r) = \{y : (x, y) \in d^{-1}([0, r])\}$ and analogously for $S_d(x, r)$. The d -diameter of a set $A \subseteq Z$ is denoted by $\text{diam}_d(A)$.

Assume that $(Z, \mathcal{B}_Z, \mu, T)$ is a dynamical system and that d is a Borel pseudometric on Z . For $x \in Z$, $n \in \mathbb{N}$ and $r \geq 0$ define the *correlation sum*

$$C_d(x, n, r) = \frac{1}{n^2} \text{card}\{(i, j) : 0 \leq i, j < n, d(T^i(x), T^j(x)) \leq r\}$$

and the *correlation integral*

$$c_d(r) = \mu \times \mu\{(x, y) : d(x, y) \leq r\} = \int_Z \mu B_d(x, r) d\mu(x).$$

Recall that c_d is non-decreasing, right continuous and tends to 1 if $r \rightarrow \infty$. Further, c_d is continuous at r if and only if $\mu S_d(x, r) = 0$ for μ -a.e. $x \in Z$, see e.g. [13, Remark 2.2].

The strong law for the correlation sum was studied under different conditions in [14, 13, 1, 15, 11]. Though not stated in this form, the following theorem was proved in [11].

Theorem 19. *Let Z be a topological space, μ be a Borel probability on Z and $T : Z \rightarrow Z$ be a μ -ergodic Borel map. Let d be a separable Borel pseudometric on Z . Then, for μ -a.e. $x \in Z$ and for every $r > 0$,*

$$\lim_{n \rightarrow \infty} C_d(x, n, r) = c_d(r) > 0$$

provided c_d is continuous at r .

Let us note that this “pseudometric” version of the strong law for correlation sums cannot be directly derived from the “metric” one. Indeed, it is true that one can easily obtain a metric space from the pseudometric one by gluing together points of zero distance, as is usually done. The considered dynamical system, however, does not necessarily fit to this projection and so, in general, there is no induced system on the obtained metric space; take e.g. the case when $Z = \mathbb{R}^\infty$, T is the shift and $d(x_0^\infty, y_0^\infty) = |x_0 - y_0|$.

The proof from [11], however, perfectly fits to this general setting, as was noted by the authors. Indeed, it is based on the Birkhoff ergodic theorem and on the existence of finite Borel partitions $\mathcal{A}^m = \{A_j^m : 0 \leq j \leq M_m\}$ ($m \geq 1$) with $\mu(A_0^m) \leq 2^{-m}$ and $\text{diam}_d(A_j^m) \leq 2^{-m}$ for every $j \geq 1$. Since the former is true for arbitrary ergodic system and the latter immediately follows from separability of (Z, d) and Borel measurability of d -balls, the convergence in Theorem 19 can be proved using the same reasoning as in [11]. Finally, the fact that $c_d(r) > 0$ for every $r > 0$ is obvious due to separability of (Z, d) . (To see it, take any d -ball B with radius $r/2$ and $\mu(B) > 0$ and use that $c_d(r) \geq \mu(B)^2$.)

Next we show how Theorem 3 can be obtained from Theorem 19.

Proof of Theorem 3. Let (S, ϱ) be a separable metric space, $m, k \geq 1$ be integers and $r > 0$ be such that c_k^m is continuous at it. Let ϱ^m be a metric on S^m compatible with the product topology. Put $Z = S^\infty$ and define $d = d_k^m$ by (5). Then obviously d is a continuous pseudometric on Z ; it is separable due to separability of Z .

Let X_0^∞ be an S -valued ergodic stationary process with distribution μ . We may assume that X_0^∞ is given by its Kolmogorov representation, that is, $X_n = \pi \circ T^n$, where $T : Z \rightarrow Z$ is the shift and $\pi : Z \rightarrow S$ is the projection $x_0^\infty \mapsto x_0$. Then, for every $x = x_0^\infty \in Z$ and every n , $T^n(x) = x_n^\infty$ and so

$$c_k^m(r) = c_d(r) \quad \text{and} \quad C_k^m(x, n, r) = \left(\frac{n - k + 1}{n} \right)^2 C_d(x, n - k + 1, r).$$

Application of Theorem 19 to the ergodic system $(Z, \mathcal{B}_Z, \mu, T)$ gives the desired result. \square

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